week 2.

Recall : What is $\mathbb{R}$??

algebraically : satisfies

(a1) – (a4) and (d)

(M1) – (M4)

axiom

Ordering properties : $\exists \; \mathbb{P} \subseteq \mathbb{R}$, called the set of +ve number s.t

① $a+b \in \mathbb{P}$ if $a, b \in \mathbb{P}$

② $a \cdot b \in \mathbb{P}$ if $a, b \in \mathbb{P}$

③ $\mathbb{R} = \mathbb{P} \cup -\mathbb{P} \cup \{0\}$

where $-\mathbb{P} = \{ -a \mid a \in \mathbb{P} \}$

completeness (final axiom)

$\forall \; S \neq \phi \subseteq \mathbb{R}$ which is bdd from above,

then $\exists \; m \in \mathbb{R}$ s.t. ① $s \leq m \quad \forall s \in S$

② $\forall \varepsilon > 0, \exists \; s_\varepsilon \in S$ s.t.

$m < s_\varepsilon + \varepsilon$.

In this case, m is unique and is denoted by $\sup S$.

$(\mathbb{R}, +, \cdot)$ is a $\boxed{\text{complete}}$ ordered field.

Morally, completeness "$\Leftrightarrow$" fill the elements between all rational no.

Some consequence of completeness :

① Archimedean property :

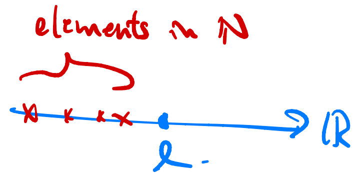
if $x \in \mathbb{R}$, then $\exists\, n_x \in \mathbb{N}$ s.t. $x < n_x$.

pf: Suppose not , $\exists\, x_0 \in \mathbb{R}$ s.t.

$\forall n \in \mathbb{N}$, $x_0 \geq n$.

$\Rightarrow$ $\mathbb{N} \subseteq \mathbb{R}$ s.t. $\mathbb{N}$ is bdd from above.

$\Rightarrow$ $\exists\ l \in \mathbb{R}$ s.t. $l = \sup \mathbb{N}$.



☆ l might not be inside $\mathbb{N}$.

let $\varepsilon = 1 > 0$, $\exists\ n_1 \in \mathbb{N}$ s.t.

$$n_1 \leq l < n_1 + 1$$

But induction property of $\mathbb{N}$ $\Rightarrow$ $n_1 + 1 \in \mathbb{N}$

$\Rightarrow$ $n_1+1 \leq l < n_1+1$ $\rightarrow\leftarrow$ #

Application of Archimedean property

Example : $\inf \{ \frac{1}{n} \mid n \in \mathbb{N} \} = 0$.

pf: ① clearly , $0 \leq \frac{1}{n}\ \forall n$. $\Rightarrow$ $0 =$ lower bdd

② $\forall \varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \frac{1}{\varepsilon}$ (A.P.)

$$\Rightarrow \quad \varepsilon > \frac{1}{n_\varepsilon}$$

i.e. $\forall \varepsilon > 0$, $\exists\, n_\varepsilon^{-1} \in \{\frac{1}{m} \mid m \in \mathbb{N}\}$ s.t.

$$n_\varepsilon^{-1} < \varepsilon + 0$$

$$\therefore \quad 0 = \inf \{\tfrac{1}{n} \mid n \in \mathbb{N}\}. \ \#$$

Corollary: if $x > 0$, then $\exists\, n_x \in \mathbb{N}$ s.t.

$$n_x - 1 \leq x < n_x.$$

pf: Consider $S = \{ m \in \mathbb{N} \mid x < m\} \subseteq \mathbb{N}$

① AP $\Rightarrow$ $S \neq \phi$.

② well-ordering $\Rightarrow$ $\exists\, m_0 \in \mathbb{N}$ st. $m_0 = \min S$ $(= \inf S)$

In particular $m_0 - 1 \notin S$

$$\Rightarrow \quad m_0 - 1 \leq x < m_0. \ \#.$$

"Completeness" $\leftarrow$ what does it mean??

Eg. Back to $\sqrt{2}$: the real +ve number solving $x^2 = 2$,

numerically,

$$\sqrt{2} = 1.\overbrace{414\,21}^{\text{non-repeating, } \infty \text{ many} \ldots}356 \ldots\ldots$$

i.e. $\sqrt{2}$ = Limit of $a_1 = 1$

??

$a_2 = 1.4$

$a_3 = 1.41$ ..... etc.

these are all rational numbers !!

very familiar !!

Expectation, all real numbers behave like this.

Mathematically :

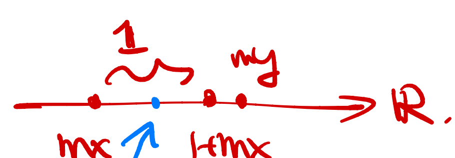
Thm (density of $\mathbb{Q}$) : $\forall x < y \in \mathbb{R}$, $\exists q \in \mathbb{Q}$ s.t.

$$x < q < y.$$

pf: If $y - x > 0$, then $\exists m \in \mathbb{N}$ s.t.

$$y - x > \frac{1}{m} \quad (\because \inf\{n^{-1} \mid n \in \mathbb{N}\} = 0)$$

$$\Rightarrow \quad my > 1 + mx.$$



Corollary above $\Rightarrow \exists n \in \mathbb{N}$ s.t.

$$n - 1 \leq mx < n$$

$$\Rightarrow \quad mx < n \leq mx + 1 < my$$

$$\Rightarrow \quad x < \frac{n}{m} \triangleq \underset{\mathbb{Q}.}{q} < y. \quad \#.$$

<u>Meaning:</u> Say $\begin{cases} \sqrt{2} = x \in \mathbb{R} \\ \sqrt{2} - \frac{1}{n} = y \in \mathbb{R} \quad \text{where } n \in \mathbb{N} \end{cases}$

$\Rightarrow \exists\, q_n \in \mathbb{Q}$ s.t. $\sqrt{2} - \frac{1}{n} < q_n < \sqrt{2}$.

$\Rightarrow q_n$ is approaching $\sqrt{2}$ as $n \to \infty$.

will make more precise later.

- this is an example of our numerical approximation of $\sqrt{2}$.

<u>Similarly</u>: Thm (density of $\mathbb{Q}^c$): $\forall\, x, y \in \mathbb{R}$ with $x < y$, $\exists\, r \in \mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}^c$ s.t. $x < r < y$.

pf: $\exists\, q \in \mathbb{Q}$ s.t. $\sqrt{2}x < q < \sqrt{2}y$ (By density of $\mathbb{Q}$!)

$$\Rightarrow \quad x < \frac{q}{\sqrt{2}} < y$$

where $r = \boxed{\frac{q}{\sqrt{2}}}$ cannot be rational.

" $q \cdot (\sqrt{2})^{-1}$ leave $\mathbb{Q}$ "

Informally,

$$\boxed{\mathbb{R} = \{\underset{\text{justify later.}}{\underline{\text{limit}}} \text{ of } \mathbb{Q}\}.}$$

Intervals: "connected" subset in $\mathbb{R}$.

$[a,b]$, $[a,b)$, (a,b), $(a,b]$, etc .....

Thm (Characterization of intervals)

If $S \subseteq \mathbb{R}$ is s.t. ① $\exists\ a \neq b \in \mathbb{R}$ s.t. $a, b \in S$

② $\forall\ x, y \in S$, $[x, y] \subseteq S$

then S = interval.

pf: "If this is true, what can S be?".

① $\Rightarrow$ $S \neq \emptyset$.

Case a: Suppose S is unbdd from both sides

Claim: $S = \mathbb{R}$.

pf of claim: let $x \in \mathbb{R}$, unbdd of S

$\Downarrow$

$\exists\ s_1, s_2 \in S$ s.t. $s_1 < x < s_2$

② $\Rightarrow$ $[s_1, s_2] \subseteq S \Rightarrow x \in S$

$\therefore\ \mathbb{R} \subseteq S \subseteq \mathbb{R}$ #

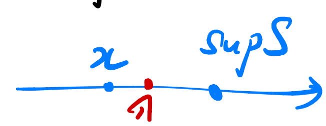
case b: Suppose S is bdd from above, but not below.

Claim: $S =$ either $(-\infty, a]$ or $(-\infty, a)$

- what is a ??
- a must be $\sup S = a$. ← exists thanks to completeness

pf of Claim: Let $x \in (-\infty, \sup S)$

then $x < \sup S$



taking $\varepsilon = \sup S - x > 0$,

$\exists\, s_0 \in S$ s.t. $\sup S < s_0 + \varepsilon = s_0 + \sup S - x$

$\Rightarrow \quad x < s_0 \leq \sup S$.

unbdd below of $S \Rightarrow \exists\, s_1 \in S$ s.t. $x > s_1$

(2) $\Rightarrow x \in [s_1, s_0] \subseteq S$

i.e. $(-\infty, \sup S) \subseteq S \overset{\text{def.}}{\subseteq} (-\infty, \sup S]$ #

similarly

case c: S is bdd from below but not above

$\Rightarrow (\inf S, +\infty) \subseteq S \subseteq [\inf S, +\infty)$ #

case d: S is bdd from above and below

$\Rightarrow (\inf S, \sup S) \subseteq S \subseteq [\inf S, \sup S]$ #

Sequence of intervals : $\{I_n\}_{n\geq 1}^{\infty}$ where I_n = intervals

special type : $\{I_n\}$ is nested if $\forall n \in \mathbb{N}$
$I_{n+1} \subseteq I_n$. ("non-increasing")

eg: $\left\{ I_n = (0, \frac{1}{n}) \right\}_{n\geq 1}^{\infty}$, $\left\{ K_n = [0, \frac{1}{n}] \right\}_{n\geq 1}^{\infty}$

$\left\{ J_n = (0, \frac{1}{n}] \right\}_{n\geq 1}^{\infty}$

Q : which one behave the last ??

- $\bigcap_{n\geq 1}^{\infty} I_n = \bigcap_{n\geq 1}^{\infty} (0, \frac{1}{n}) = \phi$

since otherwise $\exists\ x \in (0, \frac{1}{n})\ \forall n \in \mathbb{N}$

But AP $\Rightarrow$ $x > \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$

So such x does not exist.

- $\bigcap_{n\geq 1}^{\infty} J_n = \phi$ by same reasoning.

- $\bigcap_{n\geq 1}^{\infty} K_n = 0$ since if $x \in K_n\ \forall n \in \mathbb{N}$

$\Rightarrow$ $0 \leq x \leq \frac{1}{n}$.

But above reasoning $\Rightarrow$ $x \leq 0$ $\Rightarrow$ $x = 0$ #

$\boxed{\text{Upshot : Closed interval "behaves" better in above sense.}}$

Example : $\{I_n = [n, +\infty)\}_{n\geq 1}^{\infty}$ is closed but unbdd interval.

$\Rightarrow \bigcap_{n\geq 1}^{\infty} I_n = \phi$ since otherwise $\exists\, x \in \mathbb{R}$

s.t. $x \geq n \quad \forall n \in \mathbb{N}$

$\Rightarrow$ Contradicts with AP. $\cancel{A}$

$\boxed{\text{Upshot : Bdd intervals are "better"!}}$

Thm (Nested interval thm)

Suppose $\{I_n = [a_n, b_n]\}_{n\geq 1}^{\infty}$ is a seq of nested intervals which are closed and bdd,

then $\bigcap_{n\geq 1}^{\infty} I_n \neq \phi$.

Moreover, if $\inf\{b_n - a_n\} = 0$, then $\left|\bigcap_{n\geq 1}^{\infty} I_n\right| = 1$.

pf: nested intervals $\Rightarrow \begin{cases} a_{n+1} \geq a_n, & \forall n \in \mathbb{N} \\ b_{n+1} \leq b_n, & \forall n \in \mathbb{N}. \end{cases}$

$A \triangleq \{a_n \mid n \in \mathbb{N}\}, \quad B \triangleq \{b_n \mid n \in \mathbb{N}\}$

then A, B are non-empty bdd sets

$\Rightarrow$ $\underset{\underset{a}{\|}}{\sup A}$, $\underset{\underset{b}{\|}}{\inf B}$ exists by completeness.

where $\begin{cases} a_n \le a, b_m & \forall n, m \\ b \le b_m & \forall m \end{cases}$ $\Rightarrow$ $a \le b_m \quad \forall m$

$\Rightarrow$ $a \le b$.

Claim: $[a, b] = \bigcap_{n \ge 1}^{\infty} I_n$

pf: $(\subseteq)$: let $a \le x \le b$,

construction $\Rightarrow$ $\forall m, n \in \mathbb{N}$

$$a_m \le a \le x \le b \le b_n.$$

$\Rightarrow$ $x \in I_n \quad \forall n \in \mathbb{N}$

$\Rightarrow$ $x \in \bigcap_{n \ge 1}^{\infty} I_n$

$(\supseteq)$: $x \in \bigcap_{n \ge 1}^{\infty} I_n$

$\Rightarrow$ $a_n \le x \le b_n \quad \forall n.$

$\Rightarrow$ $\begin{cases} x \le b & (\because x = \text{lower bdd of } B) \\ x \ge a & (\because x = \text{upper bdd of } A) \end{cases}$

If $\inf\{b_n - a_n\} = 0$, then

$\forall \varepsilon > 0$, $\exists\, n \in \mathbb{N}$ s.t. $b_n - a_n < \varepsilon$.

$\Rightarrow 0 \le b - a < \varepsilon$.

i.e. $\forall \varepsilon > 0$, we have $0 \le b - a < \varepsilon$.

$\Rightarrow$ $b - a = 0$ since otherwise we might take $\varepsilon = b - a > 0$ s.t. $b - a < b - a$ $\rightarrow \sigma$.

So $b - a = 0 \Rightarrow \bigcap_{n \ge 1}^{\infty} I_n = \{a = b\}$ #

Thm: $[0,1]$ is uncountable.

pf: If not, then $[0,1] = \{x_n\}_{n \ge 1}^{\infty}$.

taking $I_0 = [0,1]$, then ① $\exists$ closed $I_1 \subseteq I_0$ s.t. $x_1 \notin I_1$

② $\exists$ closed $I_2 \subseteq I_1$ s.t. $x_2 \notin I_2$

⋮

ⓚ $\exists$ closed $I_k \subseteq I_{k-1}$ s.t. $x_k \notin I_k$.

$\Rightarrow \exists \{I_n\}_{n \ge 1}^{\infty}$ nested, closed, bdd s.t.

$x_n \notin I_n$ $\forall n$.

Nested interval thm (NIT)

$$\Rightarrow \bigcap_{n\geq 1}^{\infty} I_n \neq \phi . \subseteq [0,1].$$

taking $\eta \in \bigcap_{n\geq 1}^{\infty} I_n$ s.t. $\eta = x_N$ for some N.

then $\eta = x_N \in I_N \cap I_N^c = \phi$ #.